From QCD lattice calculations to the equation of state of quark matter

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Abstract

We describe two-flavor QCD lattice data for the pressure at finite temperature and zero chemical potential within a quasiparticle model. Relying only on thermodynamic selfconsistency, the model is extended to nonzero chemical potential. The results agree with lattice calculations in the region of small chemical potential.

1 Introduction

One of the fundamental issues which triggered, and has influenced since, heavy ion physics is the question of the phase structure and the thermodynamic properties of strongly interacting matter at energy densities above $1\,\mathrm{GeV/fm}^3$. Under such conditions, exceeding the energy density in nuclei but still far away from the asymptotic regime, the coupling strength α_s is large, which makes the theoretical description of the many-body problem challenging.

In the recent past the understanding of this field has become much more particularized. The phase diagram for QCD with $n_f = 2$ massless flavors, which is the case we will consider in the following, can be briefly described as follows (we refer to [1] for a detailed review). At zero quark chemical potential, $\mu = 0$, the broken chiral symmetry of hadron matter is restored within the quark-gluon plasma, at a critical temperature $T_c \approx 170\,\mathrm{MeV}$. It is thought that this second order transition persists also for nonzero μ, thus defining a critical line, which changes to a first order transition line at the tricritical point. For small temperatures and $\mu \gtrsim \mu_c$ one anticipates a color-superconducting phase of quark matter. The value of μ_c is expected to be 100...200 MeV larger than the quark chemical potential $\mu_n = 307\,\mathrm{MeV}$ in nuclear matter. Quantitative results for large α_s can be obtained from first principles by lattice calculations which were, however, restricted to finite temperature and $\mu = 0$ until very recently. Therefore, the described picture for $\mu \neq 0$ is mainly based on general arguments combined with results from various models, including extrapolations of perturbative QCD.

As a phenomenological description of the thermodynamics of deconfined strongly interacting matter we proposed a quasiparticle model [2, 3]. Its parameters are fixed by finite temperature lattice data at $\mu = 0$. We then use the fact that within the model the thermodynamic potentials at zero chemical potential and $\mu \neq 0$ are related by thermodynamic consistency. In [3] we analyzed lattice data for $n_f = 2$ flavors [4], and $n_f = 4$ [5], which were, however, still derogated by sizable lattice artefacts which have an effect on the absolute scaling of the data. We therefore introduced a constant effective number of degrees of freedom of the quasiparticles as an additional model parameter to obtain first qualitative estimates. Later we considered in [6] the lattice data [7], where also the physical case of (2+1) flavors was simulated. As the absolute scaling of the lattice data enters as an important information in particular near T_c, we applied the continuum extrapolation of the data, which was proposed in [7] for $T > 2\,T_c$, also for smaller temperatures. The results of this prescription can now be compared to new lattice data [8]. Meanwhile, there are other lattice calculations which allow to test directly the assumptions underlying the quasiparticle model as well as, for the first time, some of its predictions for nonzero chemical potential.

We will therefore consider here the presently available lattice data for $n_f = 2$. Based on that, we will fit and discuss the quasiparticle parameters at $\mu = 0$ in Section 3. In Section 4, we will briefly summarize how to extend the model to nonzero chemical potential, and compare our findings with the results [9] from lattice simulations studying the region of small μ. Section 5 concludes with the discussion of some physical implications.

2 Finite temperature lattice data

The simulations [8] are performed on lattices with spatial extent $N_\sigma = 16$ and temporal sizes $N_\tau = 4$ and $N_\tau = 6$, with an improved Wilson quark action and renormalized quark masses corresponding to fixed ratios m_{ps}/m_v of the pseudoscalar to vector meson masses. We first consider the data for two light flavors, corresponding to $0.6 \leq m_{ps}/m_v \leq 0.75$. Although this is larger than the physical value, the results are almost insensitive to the ratio, which suggests that they are not too far from the chiral limit. As expected for the rather small lattice sizes, the results for $N_\tau = 4$ and 6 differ. However, we observe that normalizing the pressure data by $p_0^{\mathrm{cont}}/p_0^{N_\tau}$, the ratio of the free limits in the continuum and on the lattice, improves considerably the consistency between the data sets. As a matter of fact, the normalized $N_\tau = 4$ data are in agreement with the normalized $N_\tau = 6$ data after rescaling by a constant of 1.14.

This simple scaling behavior for large coupling is rather remarkable. Based on this observation we suggest the continuum estimate for the pressure shown in Fig. 1. We assume here that the normalized $N_\tau = 6$ data are already close to

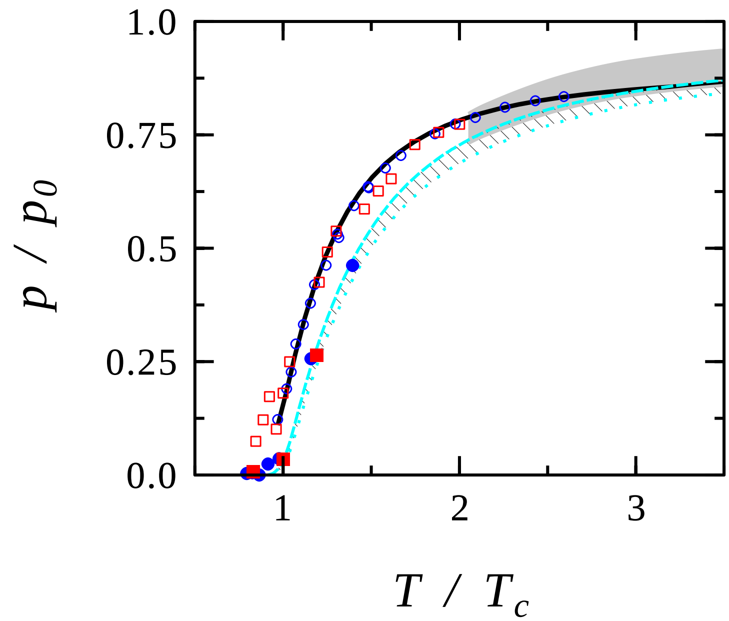

Figure 1: Compilation of $n_f = 2$ lattice data for the pressure in units of the free pressure p_0. Shown are the scaled (see text) data [8] for light quarks corresponding to meson mass ratios of $0.65 \leq m_{ps}/m_v \leq 0.75$ (small open circles: $N_\tau = 4$, small open squares: $N_\tau = 6$), and the continuum estimate [7] (grey band). The full line is the quasiparticle result. The full symbols depict the data [8] for larger quark masses, with $m_{ps}/m_v = 0.95$. The hatched band represents the SU(3) lattice data (dotted line: [10], dashed line: [11]) normalized to the corresponding free pressure.

the continuum limit. This is supported by the fact that the thus interpreted data match the aforementioned continuum estimate from the staggered quark simulations [7][1]. Therefore, a consistent picture forms for the thermodynamics of QCD with $n_f = 2$ light flavors.

In Fig. 2, the corresponding data for the entropy are shown. It is noted that since the slope of the continuum extrapolated pressure [7] is slightly larger than that from the data [8] (see Fig. 1), the upper part of the error band is already

[1] In these calculations $m_q = 0.1T$ was assumed, corresponding to $m_{ps}/m_v = 0.7$ at T_c. From the weak quark mass sensitivity observed in [8], both results should indeed be comparable.

for $T \sim 3\,T_c$ very close to the free limit. This would be in contrast to the pure gauge case, where the uncertainty due to lattice artefacts has become small, so we will assume that the lower side of this estimate is more relevant.

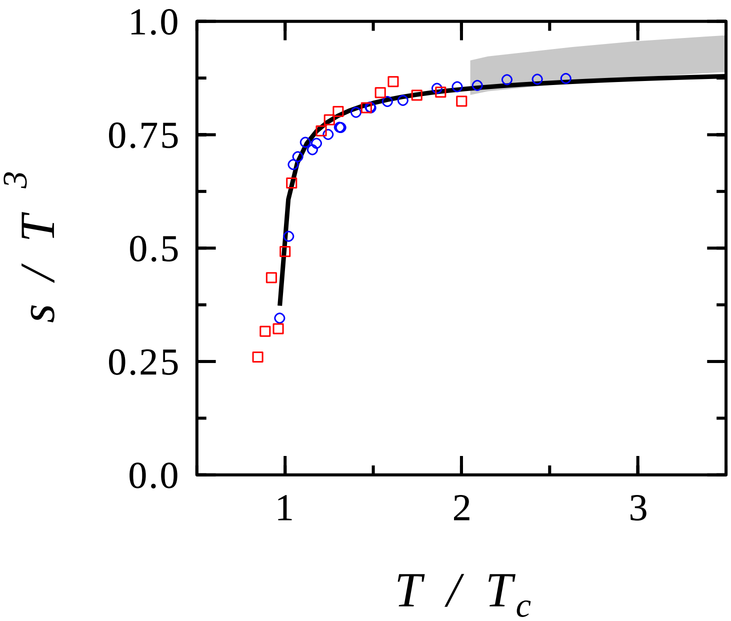

Figure 2: The lattice data for the entropy corresponding to the data for the pressure shown in Fig. 1, and the quasiparticle fit.

3 Quasiparticle model

It is trivial that the lattice data for the pressure (being smaller than the free limit) can be fitted by a gas of free particles with appropriate numbers of degrees of freedom[2] and temperature dependent effective masses. It is less obvious that these quasiparticle masses, which perturbatively are defined as the asymptotic masses $m_i = \sqrt{\alpha_s}\,\kappa_i$, $\kappa_i \propto T$, of the relevant excitations [2], i. e., their self energies at large light-like momenta, can be applied to understand thermodynamic quantities in the strong coupling regime[3]. As a matter of fact, the

[2] The degrees of freedom are the propagating excitations in the plasma (transverse gluons and the quark particle-excitations) [12]. An alternative approach with a variable number of degrees of freedom is proposed in [13].

[3] A formal reason supporting this conjecture is the stationarity of the thermodynamic potential with respect to variation of the self energies around the physical value, see [14] and the Refs. given there. Moreover, there are heuristic arguments that resummation improved

entropy shown in Fig. 2 can be described by the quasiparticle model [2] with the effective coupling[4]

$$\alpha_s(T) = \frac{12\pi}{(11N_c - 2n_f)\ln[\lambda(T-T_s)/T_c]^2} \quad \text{with} \quad \lambda = 17.1\,,\ T_s = 0.89\,T_c\,. \quad (1)$$

This is the leading order perturbative result at a momentum scale determined by the temperature: T_c/λ is related to the QCD scale parameter Λ, and T_s is an infrared regulator for the coupling. Near T_c, the resulting quasiparticle masses are large, they reach several times the value of the temperature. The existence of such heavy excitations near T_c, which we inferred from the thermodynamic bulk properties, has meanwhile been confirmed directly by lattice calculations of the propagators [16].

While in the quasiparticle model the entropy is given by the sum of the quasiparticle contributions, the pressure reads $p(T) = \sum_i p_i - B(T)$, with $p_i = \pm T \int_{k^3} \ln[1 \pm \exp\{-\omega_i/T\}]$ and $\omega_i = (m_i(T)^2 + k^2)^{1/2}$ [2, 17]. Since the derivative of the 'bag' function B is directly related to the quasiparticle masses, it is completely fixed up to an integration constant,

$$B_0 = B(T_c) = 1.1\,T_c^4\,, \quad (2)$$

which enters the fit in Fig. 1 as the third parameter of the model.

Since all the information about the coupling is encoded in the parameters T_s and λ, it is interesting to look at their flavor dependence. Comparing to the pure gauge plasma, it is recalled that in this case the pressure becomes very small close to the transition since it has to match the pressure of the heavy glue balls in the confined phase. Similarly, the entropy is small at $T \sim T_c$, which requires a large coupling there. For $n_f = 2$ the scaled entropy for $T \sim T_c$ is somewhat larger, thus close to the transition the coupling has to be smaller than for pure SU(3). However, for fixed parameters λ and T_s, the coupling (1) would increase with increasing number of active flavors. Therefore, a difference of the parameters for $n_f = 2$ to those for the pure gauge plasma [3],

$$\lambda^{\rm SU(3)} = 4.9\,, \qquad T_s^{\rm SU(3)} = 0.73\,T_c\,, \quad (3)$$

is not unexpected. Interestingly, the parameter T_s does not change by much compared to the case of $n_f = 2$.

4 Nonzero chemical potential

The quasiparticle model as applied at finite temperature in the previous section can be generalized to nonvanishing quark chemical potential μ. The quasiparticle masses now depend also on μ – explicitly by the dimensionful coefficients κ_i

leading order results might be more appropriate at large coupling than high order perturbative results [15].

[4] For the fit we considered only the normalized data [8]. The result then reproduces the extrapolated data [7] on the lower side of the estimated error band, see the remark at the end of the last section.

which are calculable in perturbation theory [12], and implicitly by the effective coupling $\alpha_s(\mu, T)$. As shown in [3], Maxwell's relation $\partial s/\partial\mu = \partial n/\partial T$, where n is the particle density $\partial p(T, \mu)/\partial\mu$ which, as the entropy, is given by the sum of the quasiparticle contributions, directly implies a partial differential equation for α_s. It is of first order and linear in the derivatives of the coupling (but nonlinear in α_s),

$$c_\mu \frac{\partial\alpha_s}{\partial\mu} + c_T \frac{\partial\alpha_s}{\partial T} = C\,, \tag{4}$$

where c_μ, c_T and C depend on μ, T and α_s. It can easily be solved by reduction to a system of coupled ordinary differential equations,

$$\frac{d\mu(s)}{ds} = c_\mu\,, \quad \frac{dT(s)}{ds} = c_T\,, \quad \frac{d\alpha_s(s)}{ds} = C\,,$$

which determines the so-called characteristic curves $\mu(s)$, $T(s)$, and the evolution of α_s along such a curve, given an initial value.

With regard to the underlying physics it is worth to point out some properties of the flow equation (4). The coefficients are combinations of products of a derivative of the quasiparticle entropy or density with respect to the quasiparticle mass, and a derivative of the quasiparticle mass with respect to μ, T or α_s. Writing down the explicit expressions, it is easy to see that the flow equation is elliptic. In particular, one finds

$$c_\mu(\mu, T=0) = 0\,, \qquad c_T(\mu=0, T=0) = 0\,.$$

Therefore, the characteristics are perpendicular to both the μ and the T axes. This guarantees that specifying the coupling on some interval on the T axis sets up a valid initial condition problem. From the temperature dependence of the effective coupling as obtained from the lattice data at $\mu = 0$, e. g. in the physically motivated parameterization (1), we can therefore determine numerically the coupling from eqn. (4), and hence the equation of state, in other parts of the μT plane.

It is instructive to consider the asymptotic limit, $\alpha_s \to 0$, of Eq. (4), where the coefficients become $c_\mu \propto \mu$, $c_T \propto T$, and $C = 0$. Then the coupling is constant along the characteristics, which become ellipses. Qualitatively, this holds also for larger coupling, see Fig. 3, so the lattice data at $\mu = 0$ are mapped in elliptic strips into the μT plane. A closer look at the characteristics emanating from the interval $[T_c, 1.06T_c]$ reveals that they intersect in a narrow half-crescent region, which indicates that there the solution of the flow equation is not unique. This, however, is only an ostensible ambiguity. It so happens that the extrapolation of the pressure becomes negative in a larger region, see Figs. 3 and 4. This implies that a transition to another phase, at a certain positive pressure, happens already outside this region, so the encountered ambiguity of the flow equation is of no physical relevance[5].

[5] We remark that the region where the solution of the flow equation is not unique is determined only by $\alpha(\mu = 0, T)$, i. e. by the parameters λ and T_s fitted from the entropy, whereas the $p = 0$ line depends also on $p(\mu = 0, T_c)$ and thus on the third parameter B_0.

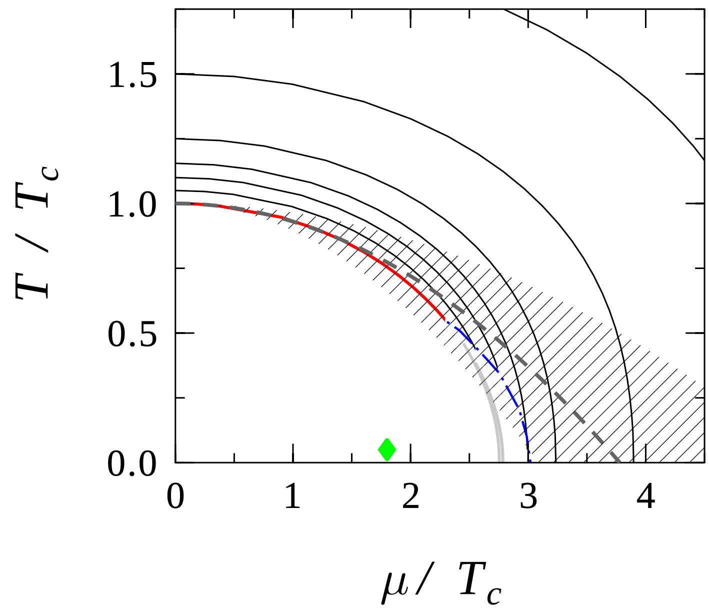

Figure 3: Represented by the full lines are the characteristics of the flow equation (4). The characteristic through T_c coincides for small μ with the critical line (with a hatched error band) obtained in the lattice calculation [9]. In the region under the dash-dotted line the resulting quasiparticle pressure is negative – a transition to another phase has to happen somewhere outside. Therefore, the narrow grey region under the $p = 0$ line, where the solution of the flow equation is not unique, is physically irrelevant. Indicated by the symbol (assuming, for the scaling, $T_c = 170\,\mathrm{MeV}$) is the chemical potential μ_n in nuclear matter.

At this point we emphasize again that this extrapolation of the quasiparticle model relies only on the requirement of thermodynamic consistency. Of course, it implicitly assumes also that the quasiparticle structure does not change, i. e., that deconfined quarks and gluons are the relevant degrees of freedom. For small enough μ and temperatures above (or near, as μ gets larger) T_c this is a justified assumption. However, the quasiparticle structure will change in the hadronic phase, when both T and μ are small, as well as for sufficiently cold and dense systems where the color-superconducting phase is expected. Although the present quasiparticle model cannot make any statements about these phases, it is interesting to observe that it 'anticipates' the existence of another phase only from the lattice input at $T > T_c$ and $\mu = 0$. An interpretation of the apparent similarity of the line of vanishing pressure in Fig. 3 with the expected transition line from the hadron to the superconducting quark matter phase, see

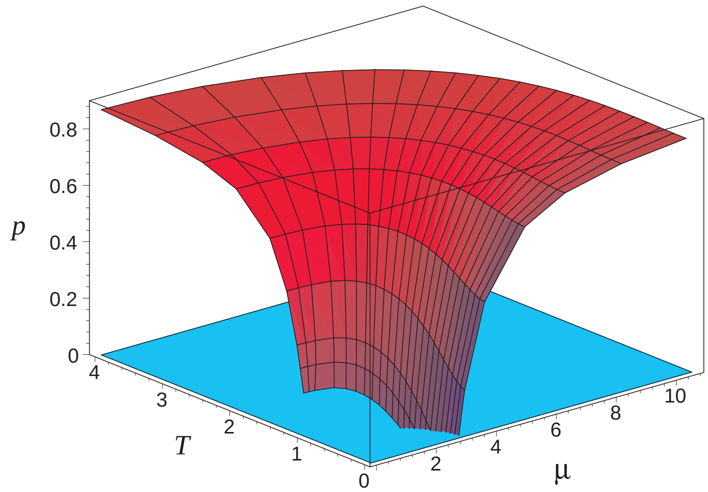

Figure 4: The pressure scaled by the free pressure $p_0(\mu, T)$; T and μ are in units of T_c. The pressure along the characteristics starting out from $T \sim T_c$ becomes negative at small T, see also Fig. 3. The change to a different phase has to happen already outside this region.

[1], remains, of course, a speculation.

There is, however, a related question which we can address with the quasiparticle model without knowing details about the other phases, just based on the fact that for nonzero chemical potential the transition from the deconfined to the confined phase occurs at the critical line $T_c(\mu)$. The critical line is expected to be perpendicular to the T axis, which has been confirmed in a recent lattice calculation [9] where also its curvature at $\mu = 0$ has been calculated[6], $T_c\, d^2T_c(\mu)/d\mu^2|_{\mu=0} \approx -0.14$. Within the quasiparticle model it is natural to relate, at least for small μ, the critical line to the characteristic through $T_c(\mu = 0)$, which, as shown above, is also perpendicular to the T axis. For small μ where only the quadratic terms are relevant (practically even for μ as large as $2\,T_c$), we indeed find the T_c characteristic in a striking agreement with the critical line from [9], see Fig. 3. Another argument supporting the above interpretation of the T_c characteristic comes from considering the case where the quark flavors have opposite chemical potentials, $\mu_u = -\mu_d = \tilde{\mu}$. With this isovector chemical potential the fermion determinant is positive definite, and standard Monte-Carlo techniques can be applied to study this system on the lattice [18]. The lattice result [9] obtained for the curvature of the critical line in that case agrees with the value quoted above for the isoscalar potential μ. Within the quasiparticle model, the equality of these two numbers is immediately evident.

[6]In passing we note the amusing fact that the result agrees with the value from the bag model assuming free massless pions for the hadronic phase.

In Ref. [9] it was furthermore mentioned that the quadratic behavior, with the same curvature as at $\mu = 0$, of the critical line is not likely to extrapolate down to small transition temperatures since $T_c(\mu)$ would then vanish at $\mu_c \sim 650\,\mathrm{MeV}$. Phenomenologically, however, μ_c is expected to be not very much larger, say at most by $200\,\mathrm{MeV}$, than the quark chemical potential $\mu_n = 307\,\mathrm{MeV}$ in nuclear matter. In the quasiparticle model, from the chemical potential where the extrapolated pressure vanishes at $T = 0$, we estimate $\mu_c \approx 3\,T_c \sim 500\,\mathrm{MeV}$.

This value is in the expected ball park, which encourages us to consider the extrapolation of the model down to smaller temperatures. Although for $T \to 0$ quark matter will be in the superconducting phase, it is still possible to give an estimate of its equation of state in that region from the quasiparticle model. The quark pairing influences thermodynamic bulk properties at the order of $(\Delta\mu)^2$, with the gap energy Δ being at most $100\,\mathrm{MeV}$ [1]. This has little effect on the energy density $e(\mu, T = 0) = \sum_i e_i(m_i; \mu, T = 0) + B(\mu, T = 0)$ as both the quasiparticle contributions and the function B are parametrically of the order $\mathcal{O}(\mu^4)$. For the pressure, however, the pairing effects become comparable to our expression $p = \sum_i p_i - B$ only when the latter gets small. Since the pressure of the thermodynamically favored superconducting phase is less than that of the plasma phase, the relation $e(p)$ as shown in Fig. 5 is therefore a lower estimate of the equation of state of cold quark matter. For $p \geq 5\,T_c^4$, we obtain

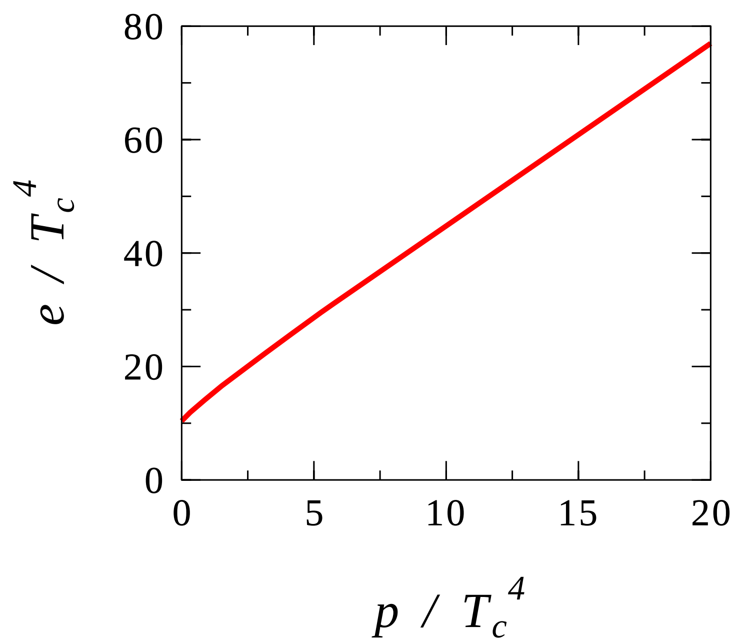

Figure 5: The estimate for the equation of state of quark matter at $T = 0$.

$e(p) \approx 13\,T_c^4 + 3.2\,p$, where the slope is mainly determined by the fact that the pressure at $T = 0$ essentially scales as μ^4. For smaller pressure, the slope is only slightly larger, and the energy at $p = 0$ is approximately[7] $11\,T_c^4$. Assuming $T_c \approx 170\,\mathrm{MeV}$, this translates into an energy density of $1\,\mathrm{GeV/fm}^3$. Bearing in mind that this is a lower estimate, and comparing to the bag model equation of state, $e^{\mathrm{bag}}(p) = 4\tilde{B} + 3p$, this result is considerably larger than estimates with commonly assumed values of the bag constant $\tilde{B}$.

Coming back to the region of the phase space where the quasiparticle model is well grounded, we finally address the question of the behavior of the pressure and the energy density along the critical line near $\mu = 0$. In the lattice simulations [9] both quantities have been found to be constant within the numerical errors. This is compatible with our result for small μ,

$$p(T_c(\mu), \mu) - p(T_c(0), 0) \approx -0.02\,\mu^2\,T_c^2\,, \tag{5}$$

the corresponding coefficient for the energy density is about three times larger. These results differ notably from the estimate from the bag model which, although the critical line has a similar shape for small μ, would yield coefficients larger by a factor of four.

5 Conclusions

Within our quasiparticle model [2, 3] we analyze recent $n_f = 2$ QCD lattice calculations [8] of the equation of state at finite temperature and $\mu = 0$. In comparison with our earlier analysis [3, 6] we find slightly changed model parameters due to now better established details of the the pressure close to T_c. We extend the quasiparticle model to finite baryon density. The resulting elliptic flow equation for the coupling relates the thermodynamic potential along the characteristic curves in the $\mu\,T$ plane. We argue that the characteristic line through $T_c(\mu = 0)$ is related to the critical line in the phase diagram. This is confirmed by comparing our results for the curvature of the critical line at $\mu = 0$, and the variation of the equation of state along it, with recent lattice simulations [9] exploring the region of small μ.

Since the quark pairing at small T does not significantly change the bulk properties of deconfined matter, we give an estimate for the equation of state of cold quark matter. Energy density and pressure are almost linearly related, as in the bag model, however with parameters calculated from the finite temperature lattice data at $\mu = 0$. The relevant physical scale is given by the transition temperature T_c, and the parameter corresponding to the bag constant turns out to be large compared to conventional estimates, $\gtrsim 250\,\mathrm{MeV}^4$.

As shown in [3], such an equation of state would allow for pure quark stars with masses $\leq 1 M_\odot$ and radii ≤ 10 km. Similar small and light quark stars have also been obtained within other approaches, cf. [19]. Such objects are of interest in the ongoing discussion of the data of the quark star candidate RXJ1856.5-3754 [20]. It should be emphasized, however, that the outermost layers of such

[7] This value is somewhat smaller than what was found previously in [6].

pure quark stars are metastable with respect to hadronic matter with a larger pressure at $\mu \sim \mu_c$. The details of the star structure depend sensitively on the hadronic equation of state [21]. However, as discussed in [22], a stable branch of hybrid stars with a dense quark core and a thin hadronic mantle could indeed be possible.

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